Invertible Matrix Worksheet Math 202

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Square Matrices

1. Suppose that A is an $n \times n$ matrix. Prove the following statements by "walking through the tree" of the Invertible Matrix Theorem. **You must show every step**.

   (a) Suppose that an $n \times n$ matrix A is invertible. Prove that the columns of A span $\mathbb{R}^n$.

   If A is invertible
   $\Rightarrow A \sim I_n$
   $\Rightarrow A$ has n pivots
   $\Rightarrow A$ has pivot in each Row
   $\Rightarrow$ columns of A span $\mathbb{R}^n$

   (b) Suppose that an $n \times n$ matrix A is not invertible. Prove that the columns of A are linearly dependent.

   If A is not invertible
   $\Rightarrow A \not\sim I_n$
   $\Rightarrow A$ doesn't have n pivots
   $\Rightarrow A$ doesn't have pivot in each column
   $\Rightarrow$ columns of A are Dependent

   (c) Suppose that A is an $n \times n$ matrix, and that $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a unique solution. Prove that A is invertible.

   If $A\vec{x} = \vec{0}$ has unique solution
   $\Rightarrow A$ [redacted] has pivot in each column
   $\Rightarrow A$ has n pivots
   $\Rightarrow A \sim I_n \Rightarrow A$ is invertible

   (d) Suppose that A is an $n \times n$ matrix, and that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ does not have a solution for some $\vec{\mathbf{b}}$ in $\mathbb{R}^n$. Prove that A is not invertible.

   $A\vec{x} = \vec{b}$ doesn't always have a solution
   $\Rightarrow$ NOT a pivot in each row
   $\Rightarrow$ NOT n pivots
   $\Rightarrow A \not\sim I_n$
   $\Rightarrow A$ NOT invertible

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2. Rewrite $(AB)^{-1}(B + A)$ using the properties of matrix operations.

   (Careful: Multiplication is *not* commutative).

$$(AB)^{-1}(B+A)$$
$$= B^{-1}A^{-1}(B+A)$$
$$= B^{-1}A^{-1}B + B^{-1}\underline{A^{-1}A}$$
$$= B^{-1}A^{-1}B + B^{-1}$$

← done! cannot go further.

3. Rewrite $(B + A)(AB)^{-1}$ using the properties of matrix operations.

   (Careful: Multiplication is *not* commutative).

$$(B+A)(AB)^{-1}$$
$$= (B+A)\cdot B^{-1}A^{-1}$$
$$= \underline{BB^{-1}}A^{-1} + AB^{-1}A^{-1}$$
$$= A^{-1} + AB^{-1}A^{-1}$$

done! ← cannot go further.

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4. Suppose that $2\vec{a} + 3\vec{b} = \vec{c}$. Prove that $A\vec{c}$ is in the span of $A\vec{a}, A\vec{b}$.

$$A\vec{c} = A(2\vec{a} + 3\vec{b})$$
$$= 2 \cdot A\vec{a} + 3 \cdot A\vec{b}$$
so $A\vec{c} \in \text{Span}\{A\vec{a}, A\vec{b}\}$

5. Suppose that the second column of B is all zero's. What can be said about the ~~third~~ second column of AB?

$$B = [\vec{b_1}, \vec{0}, \vec{b_3} \cdots \vec{b_n}]$$
so $AB = [A\vec{b_1} \quad \underbrace{A\vec{0}}_{=\vec{0}} \quad A\vec{b_3} \cdots A\vec{b_n}]$

the second column of AB is $\vec{0}$

6. Suppose the first two columns $\vec{b}_1$, and $\vec{b}_2$, of B are equal. What can be said about the columns of AB?

$$B = [\vec{b_1} \quad \vec{b_1} \quad \vec{b_3} \cdots \vec{b_n}]$$
$$AB = [A\vec{b_1} \quad A\vec{b_1} \quad A\vec{b_3} \cdots A\vec{b_n}]$$
they are equal

7. Suppose that the columns of B are linearly *dependent*. Prove that the columns of AB are linearly *dependent*.

If there is a nontrivial relation
$$c_1\vec{b_1} + \cdots + c_n\vec{b_n} = \vec{0}$$
then $A(c_1\vec{b_1} + \cdots + c_n\vec{b_n}) = A \cdot \vec{0}$
$$c_1 A\vec{b_1} + \cdots + c_n \cdot A\vec{b_n} = \vec{0}$$
this is a nontrivial dependence relation for col's of A

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Non-Square Matrices

1. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear. Prove that T cannot be one-to-one.

$P \Leftrightarrow Q$ T is one-to-one $\Leftrightarrow$ std matrix A has pivot in each column

$\neg Q$ but a 2x3 matrix A cannot have 3 pivots in only two rows

$\neg P$ Therefore T cannot be one-to-one.

2. Give an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ that *is* onto.

define $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

then $T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is onto $\mathbb{R}^2$

3. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear. Is it true that T is one-to-one if and only if T is onto? Why doesn't this violate the invertible matrix theorem?

No. we have seen above that this transformation cannot be one-to-one but it can be onto.

This does not violate the IMT because T is not defined by a square matrix.

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Definitions

set of vectors that can be gotten from $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

(a) Define Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \vec{b} \text{ in } \mathbb{R}^n \;:\; \begin{array}{l} \vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \\ \text{for some } c_1, c_2, c_3 \text{ in } \mathbb{R} \end{array} \right\}$$

(b) Define linear **Independence** of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

the set is <u>independent</u>

$\iff$

the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has <u>ONLY</u> the trivial solution

(c) Define "T is a linear transformation"

$$\left.\begin{array}{l} T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \\ T(c\cdot\vec{u}) = c\cdot T(\vec{u}) \end{array}\right\} \text{ for } \underline{\text{ALL}}\ \vec{u}, \vec{v} \text{ in } \mathbb{R}^n \text{ and all } c \text{ in } \mathbb{R}$$

(d) Define "T is one-to-one"

- $T(\vec{x}) = \vec{b}$ has <u>AT MOST</u> one solution for each $\vec{b}$

(e) Define "T is onto"

$T(\vec{x}) = \vec{b}$ has <u>AT LEAST</u> one solution for each $\vec{b}$

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Theorems

Theorem 2 The reduced echelon form of a linear system has three possible cases

NONZERO

(a) The system has <u>0</u> solutions if <u>it contains $[0 \cdots 0 \mid \blacksquare]$</u>

(b) The system has <u>1</u> solutions if <u>it has a pivot in each coeff <u>column</u></u>

(c) The system has <u>∞-many</u> solutions if <u>it has a coeff <u>column</u> <u>without</u> a pivot</u>

Theorem 4: The columns of an $m \times n$ matrix A span $\mathbb{R}^m$

if and only if there is a pivot <u>in each <u>ROW</u></u>

Shortcuts to Recognize Dependence

- *If* one column of A is a multiple of another, *then* the columns of A are linearly dependent.
- *If* $\{\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n\}$ contains $\vec{\mathbf{0}}$, *then* $\{\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n\}$ is linearly dependent.
- *If* an $m \times n$ matrix A has more columns than rows (if $n > m$), *then* the columns of A are linearly dependent.

Theorem 5 If A is an $m \times n$ matrix, $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, *Then*

- $A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = $ <u>$A\vec{u} + A\vec{v}$</u>
- $A(c \cdot \vec{\mathbf{u}}) = $ <u>$c \cdot A\vec{u}$</u>

Properties of Linear Transformations

- If T is linear, then $T(\vec{\mathbf{0}}) = $ <u>$\vec{0}$</u>
- T is linear $\iff T(c \cdot \vec{\mathbf{u}} + d \cdot \vec{\mathbf{v}}) = $ <u>$c \cdot T(\vec{u}) + d \cdot T(\vec{v})$</u>

Theorem 10 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear.

Then there is a unique $m \times n$ matrix A s.t. $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

In Fact, $A = \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}$ where $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Theorem 12 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear with standard matrix A.

(a) T is onto $\iff$ <u>The columns of A span $\mathbb{R}^m$</u> $\iff$ pivot in each <u>ROW</u>.

(b) T is one-to-one $\iff$ <u>The columns of A are linearly independent</u> $\iff$ pivot in each <u>COLUMN</u>.